ON THE MOTION AND STABILITY OF AN ELASTIC BODY WITH A CAVITY CONTAINING FLUID

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The problem of studying the motion and stability of an elastic body with cavities containing fluid is of definite theoretical and applied interest. This problem is an outgrowth of the problem of dynamics of a solid with a fluid [1].

Equation of motion of an elastic body with cavities containing an incompressible fluid and boundary conditions are deduced herein from the principle of least action. Conditions for the existence of some first integrals of the equations of system motion are mentioned. The equilibrium and stationary motion conditions are examined and a definition is given for the stability of such motion.

A theorem reducing the question of the stability of equilibrium or of stationary motion to a problem on the minimum of some functional is proved.

1. Let us consider some free elastic solid having a closed cavity filled entirely or partially with an ideal, homogeneous, incompressible fluid. Let us rigidly connect a rectangular Cartesian coordinate system $Ox_1x_2x_3$, which remains unchanged, to the elastic body in one of its steady states, say the undeformed state. The domains in the $x_1x_2x_3$ space occupied by the elastic body and the fluid at a given time will be denoted by τ_1 and τ_2, respectively, where S_i' denotes the boundary of the domain τ_i.

The surface S_1' of the elastic body consists of the exterior body surface S_1 and the cavity wall surface σ, i.e. $S_1' = S_1 + \sigma$.

The fluid surface S_2' generally consists of its free surface S and the part σ_2 of the cavity wall surface σ with which the fluid is contiguous at a given time, i.e. $S_2' = S + \sigma_2$.

The part of the surface σ with which the fluid is not contiguous at the given time is denoted by σ_1 so that $\sigma = \sigma_1 + \sigma_2$.

If the fluid fills the cavity completely, the surface S_2' coincides with the surface σ, i.e. $S_2' = \sigma = \sigma_2$ in this case, there are no surfaces S and σ_1. We shall henceforth denote the areas of the appropriate surfaces by the same symbols S, σ_2. The body and fluid densities are denoted by ρ_1 and ρ_2, where $\rho_2 = \text{const}$ because of the incompressibility and homogeneity of the fluid.

If the fluid partially fills a closed cavity, we shall consider the rest of the cavity to be a vacuum with pressure $p_0 = 0$.

We consider the elastic body and the fluid in its cavity as a single mechanical system, and we study its motion relative to some inertial coordinate system of axes $O'x_1'x_2'x_3'$.

The radius vector of some point P_ν of the system relative to the point O' is

$$\mathbf{r}_\nu' = \mathbf{r}_0' + \mathbf{r}_\nu \qquad (1.1)$$

where $\mathbf{r}'_0$ is the radius vector of the point O, $\mathbf{r}_\nu$ the radius vector of the point P_ν relative to the point O.

According to the theorem on the addition of velocities, the absolute velocity vector $\mathbf{v}_\nu$ of the point P_ν can be represented as

$$\mathbf{v}_\nu = \mathbf{v}_0 + \boldsymbol{\omega} \times \mathbf{r}_\nu + \mathbf{w}_\nu \tag{1.2}$$

where $\mathbf{v}_0$ is the velocity vector of the point O, $\boldsymbol{\omega}$ the angular velocity vector of the moving $Ox_1x_2x_3$ coordinate system, $\mathbf{w}_\nu$ the relative velocity vector of the point P_ν.

By definition $\mathbf{w}_\nu = d\mathbf{r}_\nu/dt$, where the derivative with respect to the time t is taken in the moving coordinate system. The displacement vector of points of the elastic body due to elastic deformation are denoted by $\mathbf{u}_\nu(\mathbf{r}_\nu^\circ, t)$, where $\mathbf{r}_\nu^\circ$ is the radius vector of the point P_ν of the body in its undeformed state. We assume the function $\mathbf{u}(\mathbf{r}^\circ, t)$ to be a continuously differentiable function of its arguments. The vector $\mathbf{u}_\nu$ is evidently the vector of relative displacement of the point P_ν of the body relative to the moving coordinate system $Ox_1x_2x_3$, so that for points of the elastic body

$$\mathbf{r}_\nu = \mathbf{r}_\nu^0 + \mathbf{u}_\nu, \qquad d\mathbf{u}_\nu = \mathbf{w}_\nu\, dt \tag{1.3}$$

The kinetic energy of the system is comprised of the kinetic energies of the elastic body and the fluid

$$E = {}^1/_2 \sum_\nu m_\nu v_\nu^2 = {}^1/_2 M \mathbf{v}_0^2 + M\mathbf{v}_0 \cdot (\boldsymbol{\omega} \times \mathbf{r}_c) + {}^1/_2\, \boldsymbol{\omega} \cdot \boldsymbol{\theta} \cdot \boldsymbol{\omega} +$$
$$+ \mathbf{v}_0 \cdot \Big(\int_{\tau_1} \rho_1 \mathbf{w}\, d\tau + \int_{\tau_2} \rho_2 \mathbf{w}\, d\tau \Big) + \boldsymbol{\omega} \cdot \Big(\int_{\tau_1} \rho_1 \mathbf{r} \times \mathbf{w}\, d\tau + \int_{\tau_2} \rho_2 \mathbf{r} \times \mathbf{w}\, d\tau \Big) +$$
$$+ {}^1/_2 \Big(\int_{\tau_1} \rho_1 \mathbf{w}^2\, d\tau + \int_{\tau_2} \rho_2 \mathbf{w}^2\, d\tau \Big) \tag{1.4}$$

$$M = M_1 + M_2, \qquad \mathbf{r}_c = M^{-1}(M_1\mathbf{r}_1 + M_2\mathbf{r}_2), \qquad \boldsymbol{\theta} = \boldsymbol{\theta}^{(1)} + \boldsymbol{\theta}^{(2)}$$

Here M is the system mass, equal to the sum of the elastic body mass M_1 and the fluid mass M_2, $\mathbf{r}_c$ is the radius vector of the system center of mass relative to the point O, where $\mathbf{r}_1$ and $\mathbf{r}_2$ are radius vectors of the body and fluid centers of mass, $\boldsymbol{\theta}^{(1)}$ and $\boldsymbol{\theta}^{(2)}$ the inertia tensors of the body and fluid, $\boldsymbol{\theta}$ the system inertia tensor for the point O, the subscript ν denotes summation over all points of the system.

It is easy to see [1] that the system momentum and moment of momentum vectors relative to the point O are respectively

$$\mathbf{Q} = \sum m_\nu \mathbf{v}_\nu = \mathrm{grad}_{\mathbf{v}_0} E, \qquad \mathbf{G} = \sum_\nu \mathbf{r}_\nu \times m_\nu \mathbf{v}_\nu = \mathrm{grad}_\omega E \tag{1.5}$$

The equation of motion of an elastic body with a fluid can be deduced from the principle of least action in the Hamilton-Ostrogradskii form, just as is the equation of motion of a solid with a fluid [1]. The difference would just be in taking account of the relative motion of the points of the elastic body relative to the $Ox_1x_2x_3$ coordinate system, and the internal stresses originating therein, whose density in an area with exterior normal $\mathbf{n}$ (relative to the considered part of the elastic body) will be denoted by $\mathbf{p}_n$. As is known [2], the vector $\mathbf{p}_n$ is expressed linearly in terms of the stresses $\mathbf{p}_i (i = 1,2,3)$ on areas taken at the same point of the elastic body, which are orthogonal to the x_i-axes

$$\mathbf{p}_n = \mathbf{p}_1 n_1 + \mathbf{p}_2 n_2 + \mathbf{p}_3 n_3 \tag{1.6}$$

where n_i are cosines of angles formed by the unit external normal vector $\mathbf{n}$ and the

x_i-axes. We denote the projections of the vector $\mathbf{p}_i$ on the axis x_j by p_{ij}, where $p_{ij} = p_{ji}$.

From the analytical mechanics viewpoint, the internal stresses are the reactions of couplings existing between points of the elastic body under the condition of continuity of the displacement field $\mathbf{u}(\mathbf{r}^\circ, t)$. Utilizing the release principle, we include the stresses among the active forces by hence considering the possible displacements $\delta\mathbf{r}$ of points of the elastic body relative to the $Ox_1x_2x_3$ coordinate system as perfectly arbitrary continuous functions of the coordinates x_i of points of the body. The sum of the elementary work of the internal surface stress forces on the possible displacements $\delta\mathbf{r}$ [2, 3]

$$-\int_{\tau_1}\left(\mathbf{p}_1\cdot\frac{\partial\delta\mathbf{r}}{\partial x_1}+\mathbf{p}_2\cdot\frac{\partial\delta\mathbf{r}}{\partial x_2}+\mathbf{p}_3\cdot\frac{\partial\delta\mathbf{r}}{\partial x_3}\right)d\tau \qquad (1.7)$$

should hence be included in the work of the active forces in the expression of the Hamilton-Ostrogradskii principle.

Performing the appropriate computations, analogous to those on pp. 31-35 in [1], we obtain the following equations of motion of an elastic body with a fluid in its cavity from the principle of least action

$$\frac{d\mathbf{Q}}{dt}+\boldsymbol{\omega}\times\mathbf{Q}=\mathbf{K} \qquad \left(\mathbf{K}=\sum_\nu \mathbf{F}_\nu\right) \qquad (1.8)$$

$$\frac{d\mathbf{G}}{dt}+\boldsymbol{\omega}\times\mathbf{G}+\mathbf{v}_0\times\mathbf{Q}=\mathbf{L} \qquad \left(\mathbf{L}=\sum_\nu \mathbf{r}_\nu\times\mathbf{F}_\nu\right) \qquad (1.9)$$

$$\rho_1\left(\frac{d\mathbf{v}}{dt}+\boldsymbol{\omega}\times\mathbf{v}\right)=\rho_1\mathbf{F}+\frac{\partial\mathbf{p}_1}{\partial x_1}+\frac{\partial\mathbf{p}_2}{\partial x_2}+\frac{\partial\mathbf{p}_3}{\partial x_3} \qquad (1.10)$$

$$\frac{d\mathbf{v}}{dt}+\boldsymbol{\omega}\times\mathbf{v}=\mathbf{F}-\frac{1}{\rho_2}\operatorname{grad}p \qquad (1.11)$$

and the dynamic boundary conditions

$$\begin{gathered}\mathbf{p}_n=\mathbf{F}_n \text{ on } S_1,\ \mathbf{p}_n=0 \text{ on } \sigma_1\\ \mathbf{p}_n=(p-2H\alpha_1)\,\mathbf{n}^{(2)} \text{ on } \sigma_2\\ p=2H\alpha \text{ on } S,\ \cos\theta=-\alpha_1/\alpha \text{ on } f\end{gathered} \qquad (1.12)$$

Here p is the hydrodynamic pressure, $\mathbf{K}$ and $\mathbf{L}$ the principal vector and principal moment of the external active forces applied to the system, with respect to the point O; $\mathbf{F}$ and $\mathbf{F}_n$ are densities of the mass and external surface forces acting on the elastic body and fluid, α and α_1 surface tension coefficients on the fluid vacuum and fluid-elastic body surfaces, $2H$ the mean curvature of the fluid surface S_2', θ the boundary angle on the line f of intersection between the free fluid surface S and the cavity walls σ, $\mathbf{n}^{(i)}$ the unit external vector normal to the boundary S_i' of the domain τ_i $(i = 1,2)$. Equations (1.8) and (1.9) express the general dynamics theorems of the system momentum and moment of momentum; (1.10) are the equations of motion of a continuum, which take the form of the Euler equations (1.11) for a perfect fluid. The equation of fluid incompressibility

$$\operatorname{div}\mathbf{w}=0 \qquad (1.13)$$

as well as definite kinematic conditions [2] on the boundaries of the domains τ_i should be appended to these equations. Let us note that (1.10) and (1.11) are expediently written in the relative velocities $\mathbf{w}$ in a number of cases, by expressing the absolute velocities $\mathbf{v}$ according to (1.2).

The obtained system of equations of motion (1.8)–(1.11), (1.13) is an open system of nonlinear equations. To close it, it is necessary to append relationships expressing the stress-strain dependencies, which are connected with the selection of a definite continuum model. Let us take the model of a solid deformable body, considered as a material continuum, for which the strain processes are reversible. To obtain a closed system of equations in this case, it is sufficient to give the external heat influx dq and the internal energy U or the free energy $A = U - Ts$ referred to unit body mass, as is known [2]. We shall henceforth assume that the densities of the internal energy U or the free energy A are defined completely by the strains and entropies s or the absolute temperature T, i.e. by the following functions:

$$U = U(\varepsilon_{ij}, s), \; A = A(\varepsilon_{ij}, T) \tag{1.14}$$

where

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j^\circ} + \frac{\partial u_j}{\partial x_i^\circ} + \sum_{k=1}^{3} \frac{\partial u_k}{\partial x_i^\circ}\frac{\partial u_k}{\partial x_j^\circ}\right) \qquad (i, j = 1, 2, 3) \tag{1.15}$$

denote the finite strain tensor components.

The stress tensor components are expressed in terms of the strain and entropy or temperature by the following equations of state:

$$p_{ij} = \rho_1 \frac{\partial U}{\partial \varepsilon_{ij}} = \rho_1 \frac{\partial A}{\partial \varepsilon_{ij}} \tag{1.16}$$

where the entropy s or temperature T must be determined by utilizing the thermodynamic equations

$$T = \frac{\partial U}{\partial s} \quad \text{or} \quad s = -\frac{\partial A}{\partial T} \tag{1.17}$$

and the equations of external heat influx

$$dq = T\, ds \tag{1.18}$$

expressing the second law of thermodynamics.

The right side of this equation can be expressed by taking account of (1.17) or in terms of U and s, or A and T. The adiabatic processes $dq = ds = 0$, consequently it is convenient to utilize the internal energy U. For isothermal processes $T = \text{const}$, hence it is convenient to utilize the free energy A; Eq. (1.18) is used to determine the heat infulx.

Moreover, it is necessary to append the continuity equation of an elastic body

$$\partial\rho_1/\partial t + div(\rho_1 \mathbf{w}) = 0 \tag{1.19}$$

utilized to determine the body density $\rho_1(x_1, x_2, x_3, t)$, and also the differential equations for some variable parameters when the forces acting on the system depend thereon [1].

The dynamic equations (1.8)–(1.11), the continuity equations (1.13) and (1.19), the equations of state (1.16) taking account of (1.15) and (1.17), and the heat influx equation (1.18) form, together with the boundary conditions and the equations for the parameters, a complete closed system of nonlinear equations of motion of an elastic body with a cavity containing a fluid.

Let us note that this system of equations can serve as the initial one for various approximate equations obtained by means of linearization.

2. Under specific conditions the equations of motion of an elastic body with a fluid admit of certain first integrals, of which we examine here the energy and area integrals.

Let us multiply (1.8) and (1.9) scalarly by $\mathbf{v}_0$ and $\boldsymbol{\omega}$, respectively, (1.10) and (1.11) by $\mathbf{w}d\tau$ and $\rho_2 \mathbf{w}d\tau$, respectively, and let us integrate these latter two results over the domains τ_1 and τ_2, and then let us add the equations obtained.

Taking account of the continuity equation, we will hence have [1]

$$\frac{dE}{dt} = \mathbf{K}\cdot\mathbf{v}_0 + \mathbf{L}\cdot\boldsymbol{\omega} + \int_{\tau_1}\left(\rho_1\mathbf{F} + \frac{\partial\mathbf{p}_1}{dx_1} + \frac{\partial\mathbf{p}_2}{\partial x_2} + \frac{\partial\mathbf{p}_3}{\partial x_3}\right)\cdot\mathbf{w}\,d\tau + \\ + \rho_2\int_{\tau_2}\left(\mathbf{F} - \frac{1}{\rho_2}\operatorname{grad} p\right)\cdot\mathbf{w}\,d\tau \qquad (2.1)$$

Taking account of the boundary conditions and (1.13)–(1.18), (1.19), it is easy to obtain

$$\int_{\tau_1}\left(\frac{\partial\mathbf{p}_1}{\partial x_1} + \frac{\partial\mathbf{p}_2}{\partial x_2} + \frac{\partial\mathbf{p}_3}{\partial x_3}\right)\cdot\mathbf{w}\,d\tau = \int_{S_1}\mathbf{F}_n\cdot\mathbf{w}\,ds - \frac{d}{dt}\int_{\tau_1}\rho_1 U\,d\tau + \\ + \int_{\tau_1}\rho_1\frac{dq}{dt}\,d\tau + \int_{\sigma_2}(p - 2H\alpha_1)\,\mathbf{n}^{(2)}\cdot\mathbf{w}dS$$

$$\int_{\tau_2}\mathbf{w}\cdot\operatorname{grad} p\,d\tau = \int_{\sigma_2} p\mathbf{w}\cdot\mathbf{n}^{(2)}\,ds + \int_{S} 2H\alpha\mathbf{w}\cdot\mathbf{n}dS$$

By virtue of these relationships, (2.1) becomes

$$\frac{d}{dt}\left(E + \Pi_1 + \Pi_2\right) = \mathbf{K}\cdot\mathbf{v}_0 + \mathbf{L}\cdot\boldsymbol{\omega} + \int_{\tau_1}\rho_1\mathbf{F}\cdot\mathbf{w}\,d\tau + \\ + \int_{\tau_2}\rho_2\mathbf{F}\cdot\mathbf{w}\,d\tau + \int_{S_1}\mathbf{F}_n\cdot\mathbf{w}\,dS + \int_{\tau_1}\rho_1\frac{dq}{dt}\,d\tau \qquad (2.2)$$

$$\Pi_1 = \int_{\tau_1}\rho_1 U\,d\tau, \qquad \Pi_2 = \alpha S + \alpha_1\sigma_2$$

Here Π_1 and Π_2 are the potential strain energy and the surface tension force. Equation (2.2) expresses the theorem on the kinetic energy of a system, or the first law of thermodynamics.

Let us note that when there are no internal heat sources in the body, the total heat influx per unit time equals the total heat flux within the body through its outer surface, i.e.

$$\int_{\tau_1}\rho_1\frac{dq}{dt}\,d\tau = -\int_{S_1}\mathbf{q}\cdot\mathbf{n}\,dS \qquad (2.3)$$

where $\mathbf{q}$ denotes the heat flux vector.

If all the external forces acting on the system were potential forces, possessing a stationary force function, then

$$\mathbf{K}\cdot\mathbf{v}_0 + \mathbf{L}\cdot\boldsymbol{\omega} + \int_{\tau_1}\rho_1\mathbf{F}\cdot\mathbf{w}\,d\tau + \int_{\tau_2}\rho_2\mathbf{F}\cdot\mathbf{w}\,d\tau + \int_{S_1}\mathbf{F}_n\cdot\mathbf{w}\,dS = -\frac{d\Pi}{dt} \qquad (2.4)$$

In this equality

$$\Pi = -\int_{\tau_1}\rho_1 U_1(q_j, x_i)\,d\tau - \int_{\tau_2}\rho_2 U_2(q_j, x_i)\,d\tau - \int_{S_1} U_3(q_j, x_i)\,dS \qquad (2.5)$$

denotes the potential energy of the external forces acting on the system, which generally depends on the location of the $Ox_1x_2x_3$ coordinate system in the $O'x_1'x_2'x_3'$ space defined by the generalized coordinates q_j $(j = 1,\ldots, n)$, and on the shapes of the domains τ_1 and τ_2 and of the body surface S_1. Here $U_1(q_j, x_i)$ and $U_2(q_j, x_i)$ denote the force functions of the forces applied to the body and fluid particles, respectively, and $U_3(q_j, x_i)$ is the force function of the surface forces applied to points of the body surface S_1. Under conditions (2.3) and (2.4), Eq. (2.2) becomes

$$\frac{d}{dt}(E+V) = -\int_{S_1} \mathbf{q}\cdot\mathbf{n}\,dS \qquad (V = \Pi + \Pi_1 + \Pi_2) \tag{2.6}$$

Here V denotes the system potential energy which equals the sum of potential energies of the external forces acting on the system, the surface tension forces and the strains.

Let us note that an equation of the form (2.6) is valid under the assumptions mentioned even for a nonfree elastic body with stationary constraints [1].

Under the condition that the total heat flux through the body surface is zero, i.e.

$$\int_{S_1} \mathbf{q}\cdot\mathbf{n}\,dS = 0 \tag{2.7}$$

the energy integral

$$E + V = h = \text{const} \tag{2.8}$$

follows immediately from Eq. (2.6).

Thus, if the external forces acting on the system are potential forces, and conditions (2.3) and (2.7) are satisfied, then the sum of the system kinetic and potential energies remains constant throughout the motion.

Now, let us assume that the external forces acting on a free elastic body do not yield the moment relative to some fixed axis x_3'. Under these conditions, the projection of the system moment of momentum on this axis remains constant. Indeed, let us multiply (1.8) vectorially on the left by the vector $\mathbf{r}_0'$ and let us add to (1.9), we hence obtain the equation

$$d\mathbf{G}_{0'}/dt + \boldsymbol{\omega}\times\mathbf{G}_{0'} = \mathbf{L} + \mathbf{r}_0'\times\mathbf{K}, \quad \mathbf{G}_{0'} = \mathbf{G} + \mathbf{r}_0'\times\mathbf{Q} \tag{2.9}$$

which expresses the theorem on the moment of momentum for a fixed point O'. Let us then multiply (2.9) scalarly by the unit vector $\mathbf{i}_3'$ in the x_3' direction

$$\frac{d}{dt}(\mathbf{G}_{0'}\cdot\mathbf{i}_3') - \mathbf{G}_{0'}\cdot\left(\frac{d\mathbf{i}_3'}{dt} + \boldsymbol{\omega}\times\mathbf{i}_3'\right) = 0 \tag{2.10}$$

The vector $\mathbf{i}_3'$ satisfies the Poisson equation [1]

$$d\mathbf{i}_3'/dt + \boldsymbol{\omega}\times\mathbf{i}_3' = 0$$

We hence immediately obtain the area integral

$$\mathbf{G}_{0'}\cdot\mathbf{i}_3' = \text{const} \tag{2.11}$$

This integral holds even for a nonfree elastic body if the constraints imposed admit of rotation around the line x_3'.

3. In cases when the forces acting on the elastic body and the fluid in its cavity are potential forces, and a system potential energy V exists, a system equilibrium position can be found according to the principle of virtual displacements from the condition

$$\delta V = \delta\Pi + \delta\Pi_1 + \delta\Pi_2 = \delta Q^* \qquad \left(\delta Q^* = \int_{\tau_1} \rho_1 \delta q\, d\tau\right) \tag{3.1}$$

Here δQ^* is the total heat influx to the elastic body.

Let us assume that the elastic body is either free or constrained by some holonomic constraints explicitly independent of the time. Let $q_j (j = 1, .., n \leqslant 6)$ denote the Lagrange coordinates of the $Ox_1x_2x_3$ reference coordinate system. The system potential energy V is a functional dependent on both the coordinates q_j and the body and fluid shapes as well, i.e. on the domains τ_1 and τ_2 and their boundaries; the strain energy Π_1 and surface tension Π_2 are evidently independent of q_j.

Let us write (3.1) explicitly by subtracting the term

$$\int_{\tau_2} p \operatorname{div} \delta\mathbf{r}\, d\tau = \int_{S+\sigma_2} p\mathbf{n}^{(\cdot)}\cdot\delta\mathbf{r}\, dS - \int_{\tau_2} \operatorname{grad} p\cdot\delta\mathbf{r}\, d\tau = 0$$

from the left side. We will hence have

$$\delta V = \sum_{j=1}^{n} \frac{\partial V}{\partial q_j}\delta q_j - \int_{\tau_1}\left(\rho_1 \operatorname{grad} U_1 + \frac{\partial \mathbf{p}_1}{\partial x_1} + \frac{\partial \mathbf{p}_2}{\partial x_2} + \frac{\partial \mathbf{p}_3}{\partial x_3}\right)\cdot\delta\mathbf{r}\, d\tau -$$
$$- \int_{\tau_2} (\rho_2 \operatorname{grad} U_2 - \operatorname{grad} p)\cdot\delta\mathbf{r}\, d\tau - \int_{S_1} (\operatorname{grad} U_3 - \mathbf{p}_n)\cdot\delta\mathbf{r}\, dS -$$
$$- \int_{S} (p - 2H\alpha)\, \mathbf{n}^{(2)}\cdot\delta\mathbf{r}\, dS - \int_{\sigma_2} (p - 2H\alpha_1)\, \mathbf{n}\cdot\delta\mathbf{r}\, dS + \int_{\sigma_2} \mathbf{p}_n\cdot\delta\mathbf{r}\, dS +$$
$$+ \int_{\sigma_1} \mathbf{p}_n\cdot\delta\mathbf{r}\, dS + \int_{f} (\alpha\cos\theta + \alpha_1)\, \mathbf{n}\cdot\delta\mathbf{r}\, df + \int_{\tau_1} \rho_1 \frac{\partial U}{\partial s}\delta s\, d\tau = \delta Q^*$$

By usual means we obtain the system equilibrium equation from this equation

$$\partial V/\partial q_j = 0 \qquad (j = 1, \ldots, n)$$

$$\rho_1 \operatorname{grad} U_1 + \frac{\partial \mathbf{p}_1}{\partial x_1} + \frac{\partial \mathbf{p}_2}{\partial x_2} + \frac{\partial \mathbf{p}_3}{\partial x_3} = 0 \qquad \operatorname{grad} U_2 - \frac{1}{\rho_2}\operatorname{grad} p = 0 \qquad (3.2)$$

and the heat influx equation

$$\delta q = T\delta s$$

together with the boundary conditions (1.12). The coordinates q_j of the $Ox_1x_2x_3$ reference system in equilibrium are determined from the first group of these equations, and the displacement field of the elastic body and the pressure in the fluid in equilibrium are determined from the other groups of equations.

Let us examine the case when the body is free or constrained by stationary constraints admitting rotation of the whole system as a single solid around some fixed line x'_3, and the forces acting on the system, assumed to be potential, do not yield a moment relative to this line. Under these conditions an area integral of the form (2.11) exists

$$G_{x_3'} = k = \text{const} \qquad (3.3)$$

Let us introduce a system of coordinate axes $O'\xi_1\xi_2x_3'$ rotating aroud the x_3'-axis at some angular velocity ω. We agree to select the quantity ω so that the projection of the system moment of momentum relative to the $O'\xi_1\xi_2x_3'$ coordinate system on the x_3'-axis would be zero at any time. The total system energy condition can hence be represented as

$$E + V = E^{(1)} + \frac{1}{2}\frac{k^2}{J} + V \qquad (3.4)$$

Here $E^{(1)}$ is the system kinetic energy while it moves relative to the $O'\xi_1\xi_2x_3'$ coordinate axes, and J is the system moment of inertia with respect to the x_3'-axis.

Among the real system motions there are, under the assumptions made on the forces and constraints, uniform rotations of the whole system as a single solid around the x_3'-axis determined from the equation

$$\delta W = \delta Q^*, \quad W = \frac{1}{2}\frac{k_0^2}{J} + V, \quad k_0 = J_0\omega_0 \qquad (3.5)$$

Here W denotes the changed potential energy of the system, k_0 is a fixed value of the constant k of the area integral for uniform rotation of the whole system at the angular velocity ω_0.

Let us note that W is a functional dependent on the shape of the domains τ_1 and τ_2

and their boundaries, and on the coordinates $q_r\ (r = 1,\ldots,\ n-1)$ of the $Ox_1x_2x_3$ reference system if it is agreed that q_n denotes the angle of rotation around the x_3'-axis.

From (3. 5) we obtain the following equations of stationary motion of an elastic body with a fluid:

$$\frac{\partial W}{\partial q_r} = -\frac{1}{2}\frac{k_0^2}{J^2}\frac{\partial J}{\partial q_r} + \frac{\partial V}{\partial q_r} = 0 \qquad (r = 1,\ldots,n-1)$$

$$\rho_1\left(\operatorname{grad} U_1 + \frac{k_0^2}{J^2}\mathbf{R}\right) + \frac{\partial \mathbf{p}_1}{\partial x_1} + \frac{\partial \mathbf{p}_2}{\partial x_2} + \frac{\partial \mathbf{p}_3}{\partial x_3} = 0 \qquad (3.6)$$

$$\operatorname{grad} U_2 + \frac{k_0^2}{J^2}\mathbf{R} - \frac{1}{\rho_2}\operatorname{grad} p = 0, \qquad \delta q = \frac{\partial U}{\partial s}\delta s$$

as well as the boundary conditions (1. 12). Here $\mathbf{R}$ denotes the vector of the shortest distance between the x_3'-axis and points of the body or fluid. It is understood that the equilibrium equation (3. 2) or the stationary motion equation (3. 6) can be obtained directly from the equations of motion (1. 8)–(1. 11), but for us it is important that these equations are corollaries of conditions (3. 1) or (3. 5), which are the conditions for stationarity of the potential energy V or the changed potential energy W of the system in the case $\delta Q^* = 0$.

4. An elastic body with fluid in its cavity possesses an infinite number of degrees of freedom and it must be agreed as to what is to be understood by the stability of its motion.

Firstly, the definition given by Liapunov of stability of the fluid equilibrium mode can be taken and extended to an elastic body with a fluid. The stable equilibrium modes are hence defined as those modes for which the fluid and body modes remain very slightly different from their equilibrium modes after sufficiently small perturbations have been communicated to the fluid and body, at least until arbitrarily fine threadlike or sheetlike protuberances form on the surfaces of the fluid and body. Such protuberances may be large in a linear dimension but small in volume, and they can thereby sustain small portions of the energy.

Hence, the distances l_1 and l_2, the inclinations ∇_1 and ∇_2 of the body and fluid defined as in [1], and the inclinations Δ_1 and Δ_2 of the perturbed surfaces σ_2 and S to the unperturbed surfaces $\sigma_2^{(0)}$ and $S^{(0)}$ defined as the differences between the areas of the perturbed and unperturbed surfaces are introduced

$$\Delta_1 = \sigma_2 - \sigma_2^{(0)}, \quad \Delta_2 = S_2 - S^{(0)}$$

Another definition of the stability of the unperturbed motion can be taken by introducing some integral characteristics of continuum motion [1].

For definiteness, let us take L_2 norms as relative displacement and velocity fields at the time t by defining them by means of the equations

$$\|\mathbf{u}^{(i)}\|^2 = \frac{1}{M_i}\int_{\tau_i}\rho_i\mathbf{u}^2 d\tau, \qquad \|\mathbf{w}^{(i)}\|^2 = \frac{1}{M_i}\int_{\tau_i}\rho_i\mathbf{w}^2 d\tau \qquad (i = 1, 2)$$

D e f i n i t i o n. If for any arbitrarily assigned positive numbers A_1 and A_2, no matter how small, there can be selected positive numbers λ_1, λ_2 and λ_3 such that for any initial values q_{j0}, $\dot{q}_{j0}$, $\mathbf{w}_0$, Δ_{i0}, $\mathbf{u}_0$ (or l_{i0}) satisfying the conditions

$$|q_{j0}| \leqslant \lambda_1, \qquad |\dot{q}_{j0}| \leqslant \lambda_2, \qquad \|\mathbf{w}_0^{(i)}\| \leqslant \lambda_2 \qquad \|\mathbf{u}_0^{(i)}\| \leqslant \lambda_1 \qquad (4.1)$$
$$(|l_{i0}| \leqslant \lambda_1,\ \nabla_{i0} \geqslant \varepsilon l_{i0}), \qquad |\Delta_{i0}| \leqslant \lambda_1, \qquad |W_0| \leqslant \lambda_3$$

and for any $t \geqslant t_0$ (or at least until $\nabla_t \geqslant \varepsilon l_t$), the following inequalities would be satisfied:

$$|q_j| < A_1, \quad |q_j^{\cdot}| < A_2, \quad \|\mathbf{u}^{(i)}\| < A_1 \quad (|l_i| < A_1)$$
$$|\Delta_i| < A_1, \quad \|\mathbf{w}^{(i)}\| < A_2 \tag{4.2}$$

then the unperturbed stationary motion (or equilibrium) of an elastic body with a fluid is stable; otherwise it is unstable.

Here ε denotes some sufficiently small positive number, and εl_t can be considered as possible inclinations of the body and fluid [1]; for unperturbed motion $q_j = 0$ ($j = 1, \ldots, n-1$ for the atationary motion case, and $j = 1, \ldots, n$ for the equilibrium case); $x_i = x^0{}_i$ ($i = 1,2,3$).

We take the concept of the minimum of the functional W as in [1], which is equivalent to the concept of positive definiteness [4] of the functional $W - W^{(0)}$, where $W^{(0)} = 0$ is the value of W for unperturbed motion.

Let us first consider adiabatic processes of elastic body deformation for which $\delta Q^* = \delta s = 0$, $s = \text{const}$.

Theorem 1. If the expression

$$W = \frac{1}{2}\frac{k_0^2}{J} + V \quad (V)$$

has a minimum $W^{(0)}$ $(V^{(0)})$ for stationary motion (equilibrium) of an elastic body with a cavity filled with fluid, then the stationary motion (equilibrium) is stable for adiabatic processes of elastic body deformation.

The proof of this theorem is analogous to the proof of Theorem 2 presented below. Let us note that we obtain the Lagrange theorem in the case $k_0 = 0$.

Let us now examine the case when the heat flux vector is $\mathbf{q} = 0$ and the temperature has the constant value T_1 for unperturbed stationary motion or equilibrium of an elastic body with fluid. Let us assume that the thermal boundary conditions are such that a local increase in temperature on the body surface is the result of heat flux directed outward, i.e. the following condition is valid on the surface of the elastic body S_1:

$$(T - T_1)\mathbf{q}\cdot\mathbf{n} \geqslant 0 \tag{4.3}$$

This condition evidently includes limit cases of both the complete heat insulation of an elastic body when $\mathbf{q} = 0$, and maintenance of the body surface at the constant temperature $T = T_1$.

The vis viva equation (2.6) under the condition (4.3) can easily be transformed into the equation

$$\frac{d}{dt}\left(E + \Pi + \int_{\tau_1} \rho_1 (U - T_1 s)\, d\tau + \Pi_2\right) = \int_{\tau_1} \left(\frac{T_1}{T} - 1\right)\mathbf{q}\cdot\mathbf{n}\, dS +$$
$$+ T_1 \int_{\tau_1} \frac{1}{T^2}\,\mathbf{q}\cdot\operatorname{grad} T\, d\tau - T_1 \int_{\tau_1} \rho_1 \bar{s}^{\cdot}\, d\tau \leqslant 0 \tag{4.4}$$

whose right side is positive [4], on the basis of the Clausius-Duhem inequality and the complementary Fourier inequality for the heat conduction. As Koiter has shown

$$U(\varepsilon_{ij}, s) - T_1 s = A(\varepsilon_{ij}, T_1) + \frac{1}{2}\frac{C_\gamma^*}{T^*}(T_1 - T)^2 \tag{4.5}$$

$$T^* = T + \theta(T_1 - T) \quad (0 < \theta < 1), \qquad C_\gamma^* = -T^*\left(\frac{\partial^2 A}{\partial T^2}\right)^*$$

where C_γ^* denotes the specific heat of the body for constant strain and temperature T^*.

Introducing the rotating $O'\xi_1\xi_2x_3'$ coordinate system of Sect. 3, and taking account of (4.5), we obtain an inequality from (4.4)

It hence follows that

$$\frac{d}{dt}\left(E^{(1)}+\frac{1}{2}\frac{k^2}{J}+V_1\right)\leqslant 0$$

$$E^{(1)}+\frac{1}{2}\frac{k^2}{J}+V_1\leqslant\left(E^{(1)}+\frac{1}{2}\frac{k^2}{J}+V_1\right)_0 \tag{4.6}$$

The subscript 0 here denotes the initial value of the corresponding quantities; V_1 is the system potential energy

$$V_1=\Pi+\Pi_1+\Pi_2+\frac{1}{2}\int_{\tau_1}\frac{C_\gamma^*}{T^*}(T_1-T)^2\rho_1\,d\tau$$

in which, in contrast to the expression for V, the strain potential energy of the elastic body is taken in the form

$$\Pi_1=\int\rho_1 A\,(\varepsilon_{ij},T_1)\,d\tau$$

i.e. is the total free energy of the body for isothermal strains at the constant temperature T_1.

T h e o r e m 2. If the expression

$$W=\frac{1}{2}\frac{k_0^2}{J}+V_1\quad(V_1)$$

has a minimum for the stationary motion (equilibrium) of an elastic body with a cavity containing a fluid, then the stationary motion (equilibrium) is stable.

P r o o f. Let us derive the system from the considered stationary motion (equilibrium) by communicating some sufficiently small initial deflections and velocities to its points so that the initial value of the energy W would be sufficiently small. Left alone, the system will later move in conformity with the inequality (4.6), which we rewrite as

$$E^{(1)}+W+\frac{1}{2}\frac{k^2-k_0^2}{J}\leqslant\left(E^{(1)}+W+\frac{1}{2}\frac{k^2-k_0^2}{J}\right)_0 \tag{4.7}$$

Let us recall that k denotes the value of the constant area for the perturbed motion, and k_0 for the unperturbed motion. When the unperturbed motion is in equilibrium, then $k=k_0=0$.

Let A_1^* be some arbitrarily small positive number. Let W_1 denote the least possible value which the functional W can take if one of the coordinates q_j ($j=1,\ldots,n-1$), the inclination Δ_i, and the norm $\|\mathbf{u}^{(i)}\|$ (or the distance l_i) equals A_1 in absolute value, and the rest of these values (and the inclination ∇_i) satisfy the conditions

$$|q_j|\leqslant A_1,\quad|\Delta_i|\leqslant A_1,\quad\|\mathbf{u}^{(i)}\|\leqslant A_1\quad(|l_i|\leqslant A_1,\ \ \nabla_i\geqslant\varepsilon l_i) \tag{4.8}$$

Let us select the number A_1 so small that the following inequality would be satisfied

$$|W_1-W^{(0)}|<A_2 \tag{4.9}$$

where A_2 is an arbirarily assigned positive number. We take the initial values of q_j, Δ_i, $\|\mathbf{u}^{(i)}\|$ (or l_i) so small that they would satisfy conditions (4.8) with inequality signs such that the initial value W_0 would be less than W_1, and the initial velocities $\mathbf{w}$ of points of the system are such that the inequality

$$\frac{1}{2}(k^2-k_0^2)\left(\frac{1}{J_0}-\frac{1}{J}\right)+E_0^{(1)}+W_0<W_1 \tag{4.10}$$

would be conserved for all values which J can have upon compliance with the conditions

$$|q_j|\leqslant A_1,\qquad\|\mathbf{u}^{(i)}\|\leqslant A_1\qquad(|l_i|\leqslant A_1) \tag{4.11}$$

For such a choice of the initial conditions, according to (4.7), we shall have the inequality

$$E^{(1)}+W<W_1 \tag{4.12}$$

in all the subsequent time of the motion while the inequalities (4.11) are satisfied, it hence follows that $W < W_1$. This inequality will be satisfied at least while $|q_j|$, $|\Delta_i|$, $\|u^{(i)}\|$ (or $|l_i|$) remain not greater than A_1. But the initial values of these quantities are less than A_1 by assumption, and since they change continuously, they cannot become greater than A_1 without first becoming equal to A_1. But this latter is impossible (under the condition $\nabla_i > \varepsilon l_i$) because of (4.12). Taking account of (4.9), it follows from the inequality (4.12) that $|E^{(1)}| < A_2$, on which basis we deduce compliance with all the conditions (4.2). The theorem is proved.

Let us note that Theorems 1 and 2 remain valid even when the fluid in the cavity is viscous [1], and dissipative forces dependent on $q_i^{\cdot}$ $(j = 1,\ldots, n - 1)$ act on the elastic body. Moreover, in this case the validity of a theorem analogous to Theorems VI and VII in [1], (pp. 184-185) can be proved.

The inversion of the Lagrange theorem given by Chetaev [5], which is analogous to the proof of Theorem III in [1] (p. 178), can also be extended for an elastic body with a fluid.

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STABILITY OF THE STEADY CONVECTIVE MOTION OF A FLUID WITH A LONGITUDINAL TEMPERATURE GRADIENT

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Papers [1-5] deal in detail with the stability of steady plane-parallel convective motion between planes at different temperatures. The present paper concerns the stability of the motion which arises between parallel vertical surfaces when the transverse temperature difference is accompanied by a longitudinal (upward or downward) temperature gradient. The presence of a longitudinal temperature gradient has a marked effect on the structure of the steady motion (see [6, 7]); the character of this effect differs depending on whether heat is applied at the bottom or at the top. The effect of top heating on the stability of convective motion was investigated by the authors of [8, 9], whose results are criticized below. To our knowledge the effect of bottom heating has not been investigated.

We solved the boundary value problem for the amplitudes of the normal perturbations